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Asymptotic behaviour of the analytic solution of the differential equation $y'(t)+y(qt)=0$ as $q \to 1^-$ *

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Abstract

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The differential equation in the title has (essentially) one analytic solution $y(t, q)$. Continuing the investigations of Feldstein and Kolb on the zeros $t_i(q)$ of the solution, first we establish the relation $t_i(q)=iq^{-i+1}[1+\mathcal{O}(1/i^2)]$ as $i \to \infty$, q fixed, then we determine the asymptotic distribution of the zeros and the limit $|y(\tau/\epsilon, q)|^{\epsilon}$ as $\epsilon \to +0$, where $\epsilon = 1-q$ and τ fixed.

Keywords: Delay differential equations; singular integral equations; asymptotic behaviour.

0. Introduction

In [5] Feldstein and Kolb have investigated among others the analytic solution $y=y(t, q)$ of the delay differential equation

$$y'(t)+y(qt)=0, \quad 0<q<1. \tag{0.1}$$

The solution y has the series expansion

$$y=y(t, q)=\sum_{n=0}^{\infty}(-1)^n q^{(n^2-n)/2}\frac{t^n}{n!}. \tag{0.2}$$

Clearly, a function $cy(t, q)$, where c is constant, is again a solution of (0.1), but this will be no matter in our investigations. In [5] the authors have shown by using complex analysis that y oscillates unboundedly on the real interval $(0, \infty)$. Let the positive zeros of $y(t, q)$ be denoted by $t_i=t_i(q)$, $i=1, 2, \ldots$, in increasing order:

$$0<t_1<t_2<\cdots, \quad t_i \to \infty, \text{ as } i \to \infty; \tag{0.3}$$

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then we have

$$\begin{aligned} &(-1)^i y(t) > 0, \quad \text{on } (t_i, t_{i+1}),\ i = 0, 1, 2, \ldots, \\ &(-1)^i y'(t_i) < 0, \quad \text{for } i = 1, 2, \ldots, \end{aligned} \tag{0.4}$$

where we used the notation $t_0 = -\infty$ which is no zero of y at all (see [5, Lemma 1]). The first zero t_1 satisfies the following relation:

$$\frac{1}{e\epsilon} < t_1 < \frac{1}{\epsilon}, \quad \epsilon = 1 - q. \tag{0.5}$$

Here, the upper bound is given in [5], the lower bound is a consequence, of a comparison theorem (see [9, Theorem 39]). A direct proof of (0.5) can also be found in [11]. The sharpness of the upper bound follows from the limit case $q = 0$ when by (0.2), $y(t, 0) = 1 - t$. Also, the lower bound cannot be sharpened as we shall see in Corollary 9.

Concerning the zeros $t_i(q)$, we find in [5] the following conjectures based on computer results:

(C_1) the quotient t_{i+1}/t_i decreases as i increases,

(C_2) $\displaystyle\lim_{i\to\infty} \frac{t_{i+1} - t_i}{t_i - t_{i-1}} = \frac{1}{q}$,

(C_3) $\displaystyle\lim_{i\to\infty} \frac{t_{i+1}}{t_i} = \frac{1}{q}$.

The last two conjectures will be a consequence of our asymptotic result

$$t_i(q) = iq^{-i+1}\left[1 + \mathscr{O}\left(\frac{1}{i^2}\right)\right], \tag{0.6}$$

while the conjecture (C_1) is still open. However, the relation (0.6) supports the conjecture (C_1) for sufficiently large i's.

Our method in this paper (similarly to [5]) uses some tools from the theory of complex analysis (for details we refer to [10]). It is obvious that $y(t, q)$ given by (0.2) has an analytic extension $y(z, q)$ to the whole complex plane $\mathbb{C}$, it is an entire function and more precisely, $y(z, q)$ is of zero order since

$$\liminf_{i\to\infty} \frac{\log\left(i!q^{(-i^2+i)/2}\right)}{i \log i} = \infty, \quad \text{for any } q \in (0, 1) \text{ fixed.}$$

Using the theorem of residues we shall prove that $y(z, q)$ has only positive zeros given in (0.3). Then by virtue of Hadamard's factorization theorem we can rewrite $y(z, q)$ as

$$y(z, q) = \prod_{i=1}^{\infty} \left(1 - \frac{z}{t_i(q)}\right), \quad z \in \mathbb{C},\ q \in (0, 1), \tag{0.7}$$

and this product form serves us to find the asymptotic form as $q \to 1 - 0$.

From (0.2) we have

$$\lim_{q\to 1-0} y(t, q) = e^{-t},$$

and this limit has no zeros in contrast to (0.7). Our main result concerns with the asymptotic of the function

$$\lim_{\epsilon\to+0}\left|y\left(\frac{\tau}{\epsilon},q\right)\right|^{\epsilon},\quad \text{for } \tau \text{ fixed.} \tag{0.8}$$

It is interesting to observe that if we replace $y(t, q)$ by e^{-t} in (0.8), the limit procedure leaves e^{-t} invariant. The corresponding results on (0.8) are formulated among others in Section 1, and the proofs are given in Section 2.

1. Preliminaries and results

To obtain (0.7) we have to investigate the zeros of $y(z, q)$ first. The main result on the zeros is the following.

Theorem 1. *The function $y(z, q)$ has only simple real (positive) zeros.*

The proof of this theorem is reduced to the ones of the following lemmas.

Lemma 2. *For sufficiently small q's the relation*

$$(-1)^i y(iq^{-i}, q) > 0,\quad i = 0, 1, 2, \ldots,$$

holds.

Lemma 3. *For fixed q the function $y(z, q)$ has exactly i zeros on $|z| \leqslant iq^{-i+1/2}$, provided $i \geqslant i(q)$, where $i(q)$ is sufficiently large, and exactly one real (consequently positive) zero $\hat{t}_i(q)$ in $(i-1)q^{-i+3/2} \leqslant |z| \leqslant iq^{-i+1/2}$ such that*

$$\lim_{i\to\infty} i\left[q^{i-1}\hat{t}_i(q) - i\right] \text{ is bounded.}$$

Remark 4. Later we shall see that $\hat{t}_i(q) = t_i(q)$.

For the proof of Lemma 3 we recall the theta function (see [1])

$$\theta(x, z) = \sum_{k=-\infty}^{\infty} (-1)^k x^{k^2} z^{2k},$$

together with two Jacobian identities [1, Chapter 17]

$$\theta(x, z) = \prod_{m=1}^{\infty} (1 - x^{2m})(1 - x^{2m-1}z^2)(1 - x^{2m-1}z^{-2}), \tag{1.1}$$

$$\sum_{k=-\infty}^{\infty} (-1)^k k x^{k^2+k} = \prod_{m=1}^{\infty} (1 - x^{2m})^3. \tag{1.2}$$

We mention here that the theta functions and the Jacobian identities turned out to be very useful tools in a similar context in [3,6], too.

We need some results for the differential equation

$$y'(t)+y(t-\tau)=0, \tag{1.3}$$

where τ is constant and $\tau > 1/\mathrm{e}$. This equation has a solution of the form $y(t)=\mathrm{e}^{-\lambda t}$ if λ is a solution of the characteristic equation

$$\lambda=\mathrm{e}^{\lambda\tau}. \tag{1.4}$$

We look for the solution of (1.4) in the form $\lambda=\alpha(\tau)+\sqrt{-1}\,\omega(\tau)$. By [9, Lemma 7, p.207] there is a unique solution of (1.4) satisfying

$$0<\omega(\tau)<\frac{\pi}{\tau}. \tag{1.5}$$

Using the parameter $\mu=\tau\omega(\tau)\in(0, \pi)$ we find the representation

$$\tau=\frac{\mu}{\sin\mu}\,\mathrm{e}^{-\mu\cot\mu}, \qquad \omega=\frac{\mu}{\tau}=\sin\mu\,\mathrm{e}^{\mu\cot\mu}, \qquad \alpha=\cos\mu\,\mathrm{e}^{\mu\cot\mu}. \tag{1.6}$$

It is not difficult to show that τ in (1.6) is a strictly increasing function of μ; hence the connection between μ and τ is one-to-one, so the functions $\alpha=\alpha(\tau)$, $\omega=\omega(\tau)$ are well-defined. Now it is clear that the function

$$y(t)=\mathrm{e}^{-\alpha(\tau)t}[A\cos(\omega(\tau)t)+B\sin(\omega(\tau)t)] \tag{1.7}$$

is a solution of (1.3) with arbitrary constants A, B.

Using the relations in (1.6) we can prove the inequality

$$\tau\omega(\tau)>\sqrt{2}\sqrt{1-\frac{1}{\mathrm{e}\tau}}, \quad \text{for } \tau>\frac{1}{\mathrm{e}}. \tag{1.8}$$

Now we can formulate two results on the zeros of $y(t, q)$.

Lemma 5. *For the consecutive zeros t_i, t_{i+1}, $i=1, 2, \ldots$, of the solution $y(t, q)$, the inequalities*

$$\frac{t_i}{q}<t_{i+1}<\frac{t_i}{q}+\frac{\pi}{\omega(\epsilon t_i/q^2)}$$

hold, where $\omega(\tau)$ is defined by (1.6) *and $\epsilon t_i/q^2>1/\mathrm{e}$ by* (0.5).

Remark 6. By a result of [7] we could obtain the sharper upper bound $t_{i+1}<t_i/q+A(\epsilon t_i/q^2)$, where $A(\cdot)$ is the first positive zero of the solution $Y(t)$ of (1.3) subject to the initial condition $Y(t)=1$ on $[-\tau, 0]$. However, we do not have an analytic formula for $A(\tau)$ as we have for $\omega(\tau)$ by (1.6).

A similar result for t_1 reads as follows.

Lemma 7. *For the first zero t_1 of $y(t, q)$ the inequalities*

$$\frac{1}{\mathrm{e}\epsilon}<t_1<\frac{1}{\mathrm{e}\epsilon}+\frac{\pi}{\omega(1/(\mathrm{e}q))}$$

hold.

Remark 8. The lower bound in Lemma 7 is the same as in (0.5) and is repeated for the sake of completeness.

An interesting consequence of Lemma 7 and (1.8) is the following.

Corollary 9. *For the first zero* t_1 *of* $y(t, q)$ *the relation*

$$\lim_{\epsilon \to +0} \epsilon t_1(q) = \frac{1}{\mathrm{e}}$$

holds.

This corollary shows the sharpness of the lower bound displayed by (0.5).
A merely technical lemma is the next one stated here without any proof.

Lemma 10. *Let the function* $\varphi(t)$ *be concave on* $[a, b]$ *such that* $\varphi'(a)$ *and* $\varphi'(b)$ *are finite. Then*

$$0 \leqslant \frac{1}{b-a} \int_a^b \varphi(t)\, \mathrm{d}t - \tfrac{1}{2}[\varphi(a) + \varphi(b)] \leqslant \tfrac{1}{8}(b-a)[\varphi'(a) - \varphi'(b)].$$

The strict inequalities hold if $\varphi'' < 0$ *on* (a, b).

Let the function $\gamma_\epsilon(s)$ be defined for $s \geqslant 1/\mathrm{e}\epsilon$ by

$$\gamma_\epsilon(s) = \begin{cases} 0, & \text{if } \dfrac{1}{\mathrm{e}\epsilon} \leqslant s < t_1(q), \\ \dfrac{1}{t_{i+1}(q) - t_i(q)}, & \text{if } t_i(q) \leqslant s < t_{i+1}(q),\ i = 1, 2, \ldots . \end{cases} \tag{1.9}$$

By Lemma 5 we have the upper bound

$$\gamma_\epsilon(s) < \frac{1}{\epsilon s}. \tag{1.10}$$

This inequality provides the possibility to find a weak limit of the functions $\gamma_\epsilon(\sigma/\epsilon)$ as $\epsilon \to +0$. First we are going to show the following relation:

$$\lim_{\epsilon \to +0} \int_{1/\mathrm{e}\epsilon}^{\infty} \frac{1}{s} \gamma_\epsilon(s)\, \mathrm{d}s = 1. \tag{1.11}$$

Comparing the coefficients of the linear terms in (0.2) and in (0.7) we get

$$\sum_{i=1}^{\infty} \frac{1}{t_i} = 1; \tag{1.12}$$

hence by Lemma 10 applied to $\varphi(t) = -1/t$,

$$\begin{aligned} \left| 1 - \int_{1/\mathrm{e}\epsilon}^{\infty} \frac{1}{s} \gamma_\epsilon(s)\, \mathrm{d}s \right| &< \left| 1 - \sum_{i=1}^{\infty} \frac{1}{2} \left(\frac{1}{t_i} + \frac{1}{t_{i+1}} \right) \right| \\ &\quad + \left| \sum_{i=1}^{\infty} \frac{1}{2} \left(\frac{1}{t_i} + \frac{1}{t_{i+1}} \right) - \int_{1/\mathrm{e}\epsilon}^{\infty} \frac{1}{s} \gamma_\epsilon(s)\, \mathrm{d}s \right| \\ &< \frac{1}{2t_1} + \frac{1}{8} \sum_{i=1}^{\infty} (t_{i+1} - t_i)^2 \frac{t_i + t_{i+1}}{t_i^2 t_{i+1}^2}. \end{aligned} \tag{1.13}$$

By Lemmas 5 and 7 and (1.8) we have

$$t_{i+1}-t_i<\frac{t_i}{q}+\frac{\pi}{\omega(\epsilon t_i/q^2)}-t_i<\frac{\epsilon t_i}{q}+\frac{\pi\epsilon t_i/q^2}{\sqrt{2}\sqrt{1-q^2/(\mathrm{e}\epsilon t_i)}}<\pi\sqrt{\epsilon}\,t_i, \tag{1.14}$$

provided ϵ is small enough. By this inequality and by (1.12),

$$\sum_{i=1}^{\infty}(t_{i+1}-t_i)^2\frac{t_i+t_{i+1}}{t_i^2t_{i+1}^2}<\pi^2\epsilon\sum_{i=1}^{\infty}\frac{t_i+t_{i+1}}{t_{i+1}^2}<2\pi^2\epsilon, \tag{1.15}$$

so the estimate in (1.13) becomes

$$\left|1-\int_{1/\mathrm{e}\epsilon}^{\infty}\frac{1}{s}\gamma_\epsilon(s)\,\mathrm{d}s\right|<\tfrac{1}{2}\mathrm{e}\epsilon+\tfrac{1}{4}\pi^2\epsilon,$$

which proves (1.11).

Let the function $\Gamma_\epsilon(\sigma)$ be defined by

$$\Gamma_\epsilon(\sigma)=\begin{cases}0, & \text{for } \sigma=0,\\ \displaystyle\int_{1/\sigma\epsilon}^{\infty}\gamma_\epsilon(s)\frac{\mathrm{d}s}{s}, & \text{for } 0<\sigma\leqslant \mathrm{e}.\end{cases}$$

The functions $\Gamma_\epsilon(\sigma)$ for $0<\epsilon<1$ are continuous, strictly increasing and by (1.10),

$$0\leqslant\Gamma_\epsilon(\sigma_2)-\Gamma_\epsilon(\sigma_1)<\sigma_2-\sigma_1,\quad \text{for } 0\leqslant\sigma_1<\sigma_2<\mathrm{e},$$

i.e., they are uniformly absolutely continuous; hence there is a sequence $\{\epsilon_l\}_{l=1}^{\infty}\subset(0,1)$ and an absolutely continuous nondecreasing function $\Gamma(\sigma)$ such that

$$\lim_{l\to\infty}\epsilon_l=0,\qquad \lim_{l\to\infty}\Gamma_{\epsilon_l}(\sigma)=\Gamma(\sigma),\quad 0\leqslant\sigma\leqslant\mathrm{e},$$

and $\Gamma'(\sigma)=(\mathrm{d}/\mathrm{d}\sigma)\Gamma(\sigma)$ exists almost everywhere (a.e.) in [0, e] satisfying the inequalities $0\leqslant\Gamma'(\sigma)\leqslant 1$ a.e.

Let $g(\sigma)\in C[0,\mathrm{e}]$ be a continuous function; then it is well known (see [2, Chapter 5]) that

$$\lim_{l\to\infty}\int_0^{\mathrm{e}}g(\sigma)\,\mathrm{d}\Gamma_{\epsilon_l}(\sigma)=\int_0^{\mathrm{e}}g(\sigma)\,\mathrm{d}\Gamma(\sigma)=\int_0^{\mathrm{e}}g(\sigma)\Gamma'(\sigma)\,\mathrm{d}\sigma,$$

or by suitable substitutions,

$$\lim_{l\to\infty}\int_{1/\mathrm{e}}^{\infty}g\left(\frac{1}{\sigma}\right)\gamma_{\epsilon_l}\left(\frac{\sigma}{\epsilon_l}\right)\frac{\mathrm{d}\sigma}{\sigma}=\int_{1/\mathrm{e}}^{\infty}g\left(\frac{1}{\sigma}\right)\Gamma'\left(\frac{1}{\sigma}\right)\frac{\mathrm{d}\sigma}{\sigma^2}.$$

We can interpret the last result as

$$\gamma_{\epsilon_l}\left(\frac{\sigma}{\epsilon_l}\right)\ \to\ \frac{1}{\sigma}\Gamma'\left(\frac{1}{\sigma}\right)=\gamma(\sigma),\quad \text{a.e. in } \left[\frac{1}{\mathrm{e}},\infty\right), \tag{1.16}$$

and similarly the weak limit property

$$\lim_{l\to\infty}\int_{1/\mathrm{e}}^{\infty}\tilde{g}(\sigma)\gamma_{\epsilon_l}\left(\frac{\sigma}{\epsilon_l}\right)\mathrm{d}\sigma=\int_{1/\mathrm{e}}^{\infty}\tilde{g}(\sigma)\gamma(\sigma)\,\mathrm{d}\sigma,\quad \text{provided } \frac{1}{\sigma}\tilde{g}\left(\frac{1}{\sigma}\right)\in C[0,\mathrm{e}]. \tag{1.17}$$

Let the function $\zeta(\tau)$ be defined as

$$\zeta(\tau)=\int_{1/\mathrm{e}}^{\infty}\frac{\gamma(\sigma)}{\sigma-\tau}\,\mathrm{d}\sigma. \tag{1.18}$$

Let us observe that

$$g(\sigma)=\frac{1/\sigma}{(1/\sigma-\tau)}=\frac{1}{1-\sigma\tau}\in C[0,\mathrm{e}],\quad \text{for } \tau<1/\mathrm{e},$$

in accordance with the condition in (1.17). The main point of our paper is formulated in the next theorem.

Theorem 11. *The function* $\zeta(\tau)$ *defined by* (1.18) *satisfies the equation*

$$\zeta(\tau)=\mathrm{e}^{\tau\zeta(\tau)},\quad \text{for } 0\leqslant\tau<\frac{1}{\mathrm{e}}, \tag{1.19}$$

and also the relation

$$\lim_{l\to\infty}\left[y\left(\frac{\tau}{\epsilon_l},1-\epsilon_l\right)\right]^{\epsilon_l}=\exp\left\{-\int_0^{\tau}\zeta(\vartheta)\,\mathrm{d}\vartheta\right\}$$

holds.

We have to solve (1.19) for $0\leqslant\tau<1/\mathrm{e}$. We obtain $\zeta(0)=\zeta'(0)=1$ and one could prove in general

$$\zeta(\tau)=\sum_{n=0}^{\infty}\frac{(n+1)^{n-1}}{n!}\tau^n, \tag{1.20}$$

and thus $\zeta(\tau)$ is analytic for $|\tau|<1/\mathrm{e}$. Particularly we have $\zeta(\tau)\neq 0$ and also $\zeta(1/\mathrm{e})=\mathrm{e}$. Again by (1.19) $\lim_{\tau\to 1/\mathrm{e}-0}\zeta'(\tau)=\infty$, hence $\tau=1/\mathrm{e}$ is a singular point of $\zeta(\tau)$. The situation becomes simpler if we consider the inverse function

$$\tau=\tau(\zeta)=\frac{\log\zeta}{\zeta},$$

which is analytic in the neighbourhood of $\zeta=\mathrm{e}$ and it has the series expansion

$$\tau=\frac{1}{\mathrm{e}}-\frac{1}{2\,\mathrm{e}^3}(\zeta-\mathrm{e})^2+\cdots. \tag{1.21}$$

Hence the function $\zeta(\tau)$ can be continued analytically to $\mathbb{C}\setminus[1/\mathrm{e},\infty)$.

Let us observe that for $\tau>1/\mathrm{e}$ by (1.4)–(1.6) we have already two solutions of the equation in (1.19):

$$\zeta(\tau)=\alpha(\tau)\pm\sqrt{-1}\,\omega(\tau).$$

Due to the local behaviour of the map $\zeta\to\tau$ in the neighbourhood of $\tau=\mathrm{e}$ given by (1.21) we obtain the relation

$$\zeta(\tau\pm 0\sqrt{-1})=\alpha(\tau)\pm\sqrt{-1}\,\omega(\tau). \tag{1.22}$$

It is known that the integral equation (1.18), (1.19) can be solved by the Plemelj formulae (see [8, p.42]), which give

$$\gamma(\tau)=\frac{1}{2\pi\sqrt{-1}}\left[\zeta(\tau+0\sqrt{-1})-\zeta(\tau-0\sqrt{-1})\right],\quad \text{a.e. in } \left[\frac{1}{e},\infty\right),$$

$$\text{p.v.}\int_{1/e}^{\infty}\frac{\gamma(\sigma)}{\sigma-\tau}\,d\sigma=\tfrac{1}{2}\left[\zeta(\tau+0\sqrt{-1})+\zeta(\tau-0\sqrt{-1})\right],\quad \text{a.e. in } \left[\frac{1}{e},\infty\right),$$

where p.v. indicates the Cauchy-type principal value of the integral which is the limit

$$\lim_{\Delta\to+0}\left[\int_{1/e}^{\tau-\Delta}+\int_{\tau+\Delta}^{\infty}\right]\frac{\gamma(\sigma)}{\sigma-\tau}\,d\sigma,$$

hence (1.22) gives a.e. in $(1/e,\infty)$:

$$\gamma(\tau)=\frac{\omega(\tau)}{\pi},\qquad \text{p.v.}\int_{1/e}^{\infty}\frac{\gamma(\sigma)}{\sigma-\tau}\,d\sigma=\alpha(\tau). \tag{1.23}$$

Let us remark here that the function $\gamma(\tau)$ in (1.23) is independent of the choice of the sequence $\{\epsilon_l\}_{l=1}^{\infty}$, which has played a role in defining the limit function $\Gamma(\sigma)$ above. Hence in (1.16), (1.17) we can use the limit procedure $\epsilon\to+0$ instead of $l\to\infty$.

Now we can announce our result on the limit in (0.8).

Theorem 12. *For* $0\leqslant\tau<1/e$ *the relation*

$$\lim_{\epsilon\to+0}\log\left[y\left(\frac{\tau}{\epsilon},q\right)\right]^{\epsilon}=-\log\zeta(\tau)+\tfrac{1}{2}[\log\zeta(\tau)]^2$$

holds, where the function $\zeta(\tau)$ *is defined by* (1.20), *and for* $\tau\geqslant1/e$ *the companion relation reads*

$$\log\left|y\left(\frac{\tau}{\epsilon},q\right)\right|^{\epsilon}\;\Rightarrow\;-\mu^2-\mu\cot\mu+\frac{1}{2}\frac{\mu^2}{\sin^2\mu},$$

where $\Rightarrow$ *denotes the convergence in measure, and the connection between* τ *and* μ *is given by* (1.6).

It might be of interest to record the byproduct (1.16), (1.23) on the density of the zeros of $y(t,q)$.

Corollary 13. *The density function* $\gamma_\epsilon(s)$ *introduced by* (1.9) *satifies the relation*

$$\lim_{\epsilon\to+0}\gamma_\epsilon\left(\frac{\sigma}{\epsilon}\right)=\frac{\omega(\sigma)}{\pi},\quad \textit{for } \sigma\geqslant\frac{1}{e},$$

where the limit is understood in weak sense.

2. The proofs

This section is devoted to the proofs of the statements formulated in Section 1.

Proof of Lemma 2. By (0.2), $y(0)=1$, so the lemma is valid for $i=0$. For $i=1$ we have

$$y\left(\frac{1}{q}, q\right) = -\frac{1}{q}\left(\tfrac{1}{2} - \tfrac{5}{6}q + \tfrac{1}{24}q^3 \pm \cdots\right),$$

which is clearly negative for sufficiently small q's.

Let $i \geqslant 2$. Then

$$y(iq^{-i}, q) = (-1)^i q^{-(i^2+i)/2} \sum_{k=-i}^{\infty} (-1)^k q^{(k^2-k)/2} \alpha_{ik}, \tag{2.1}$$

where

$$\alpha_{ik} = \frac{i!\,i^k}{(i+k)!}, \quad k = -i, -i+1, \ldots, 0, 1, \ldots . \tag{2.2}$$

We shall use the following elementary properties of the sequence $\{\alpha_{ik}\}_{k=-i}^{\infty}$:

$$0 < \alpha_{i,-i} < \alpha_{i,-i+1} < \cdots < \alpha_{i,-1} = 1 = \alpha_{i0} > \alpha_{i1} > \alpha_{i2} > \cdots > 0, \tag{2.3}$$

$$1 - \frac{k(k+1)}{2i} \leqslant \alpha_{ik} \leqslant 1 - \frac{k(k+1)}{2i} + \frac{1}{3}\frac{k^4}{i^2}, \quad \text{for } |k| \leqslant \sqrt{i}, \tag{2.4}$$

$$\alpha_{ik} < \alpha_{i,-k+1}, \quad \text{for } k = 1, 2, \ldots, [\sqrt{i}]. \tag{2.5}$$

Let the sum in (2.1) be divided into three parts according to $-i \leqslant k \leqslant -i_1$, $-i_1+1 \leqslant k \leqslant i_1$, $k \geqslant i_1 + 1$ where $i_1 = [\sqrt{i}]$ and denote the corresponding sums by Σ_1, Σ_2, Σ_3, respectively. Using the Leibniz alternating property of the terms in Σ_1, Σ_3 we find

$$|\Sigma_1| < q^{(i_1^2+i_1)/2}\alpha_{i,-i_1}, \qquad |\Sigma_3| < q^{(i_1^2+i_1)/2}\alpha_{i,i_1+1}. \tag{2.6}$$

Now we claim the inequality

$$iq^{([\sqrt{i}]^2+[\sqrt{i}])/2} < \tfrac{1}{9}, \quad \text{for } i = 2, 3, \ldots \text{ and } 0 < q \leqslant \tfrac{1}{36}. \tag{2.7}$$

This inequality is reduced to

$$\left[(j+1)^2 - 1\right] q^{(j^2+j)/2} < \tfrac{1}{9}, \quad \text{for } j = 1, 2, \ldots,$$

hence it is sufficient to show that

$$(j+1)\left(\tfrac{1}{6}\right)^{(j^2+j)/2} \leqslant \tfrac{1}{3}, \quad \text{for } j = 1, 2, \ldots .$$

Let us observe that for $j=1$ we have the equality here and the expression on the left-hand side decreases as j increases, which together imply the validity of (2.7).

By (2.2)–(2.5) we obtain $\alpha_{i0} - \alpha_{i1} = 1/(i+1)$ and

$$\left|\Sigma_2 - \frac{1}{i+1}\right| < \sum_{k=2}^{\infty} q^{(k^2-k)/2}\left[\frac{k(k+1)}{2i} + \frac{1}{3}\frac{k^3}{i^2}\right] = \frac{K_1(q)}{i} + \frac{K_2(q)}{i^2},$$

where $K_1(q)$, $K_2(q)$ are analytic functions of q for $|q| < 1$ and $K_1(0) = K_2(0) = 0$. Hence by (2.1), (2.6), (2.7) it follows that

$$(-1)^i q^{(i^2+i)/2} y(iq^{-i}, q) > \frac{1}{i+1} - \frac{K_1(q)}{i} - \frac{K_2(q)}{i^2} - \frac{2}{9i} > 0, \quad \text{for } i \geqslant 2,$$

provided $K_1(q) < \frac{1}{12}$, $K_2(q) < \frac{1}{3}$. Since the small q's meet these requirements the lemma is proved. □

Proof of Lemma 3. For the sake of brevity we denote the general term of $y(z, q)$ by z_n:

$$z_n = (-1)^n q^{(n^2-n)/2} \frac{z^n}{n!}, \quad n = 0, 1, \dots .$$

The quotient of two consecutive terms is

$$\kappa_n = \frac{z_{n+1}}{z_n} = -q^n \frac{z}{n+1}, \quad n = 0, 1, \dots,$$

hence

$$|\kappa_n| \lesseqgtr 1 \quad \text{according to} \quad |z| \lesseqgtr (n+1)q^{-n}, \tag{2.8}$$

and

$$|\kappa_0| > |\kappa_1| > |\kappa_2| > \cdots . \tag{2.9}$$

Let i be a large natural number and define the ring $\mathscr{R}_i \subset \mathbb{C}$ by

$$\mathscr{R}_i = \{z: (i-1)q^{-i+1} \leqslant |z| \leqslant iq^{-i}\}.$$

Then for any natural number $i_1 \in (0, i)$ we can write $y(z, q)$ as

$$y(z, q) = y_1 + y_2 + y_3,$$

where

$$y_1 = \sum_{n=0}^{i-i_1} z_n, \qquad y_2 = \sum_{n=i-i_1+1}^{i+i_1-1} z_n, \qquad y_3 = \sum_{n=i+i_1}^{\infty} z_n.$$

Using (2.8), (2.9) we have the estimates on $\mathscr{R}_i$:

$$|y_1| \leqslant |z_{i-i_1}| \sum_{n=0}^{i-i_1} \frac{1}{\prod_{j=n}^{i-i_1-1} |\kappa_j|} < |z_{i-i_1}| \sum_{m=0}^{\infty} \frac{1}{|\kappa_{i-i_1}|^m} = \frac{|z_{i-i_1}|}{1 - 1/|\kappa_{i-i_1}|}, \tag{2.10}$$

and similarly

$$|y_3| \leqslant \frac{|z_{i+i_1}|}{1 - |\kappa_{i+i_1}|}. \tag{2.11}$$

Introducing the notation

$$z = iq^{-i+1/2}\vartheta, \tag{2.12}$$

we have

$$y_2(z, q) = z_i \sum_{k=-i_1+1}^{i_1-1} (-1)^k q^{k^2/2} \vartheta^k \alpha_{ik} = z_i S_1, \tag{2.13}$$

where α_{ik} was introduced in (2.2). Using (2.4), we obtain

$$\left|S_1 - \sum_{k=-i_1+1}^{i_1-1} (-1)^k q^{k^2/2} \vartheta^k \left(1 - \frac{k(k+1)}{2i}\right)\right| < \sum_{k=-i_1+1}^{i_1-1} q^{k^2/2} |\vartheta|^k \frac{k^4}{3i^2}$$
$$< \frac{1}{3i^2} \sum_{k=-\infty}^{\infty} k^4 q^{k^2/2} |\vartheta|^k = \frac{S_2}{i^2},$$

consequently by (2.2),

$$\left|S_1 - \theta(\sqrt{q}, \sqrt{\vartheta}) + \frac{1}{2i}\tilde{\theta}(\sqrt{q}, \sqrt{\vartheta})\right| < \frac{S_2}{i^2} + S_3 + \frac{S_4}{i},$$

where the theta function $\theta(x, z)$ was introduced in Section 1 and the function $\tilde{\theta}(x, z)$ is defined as

$$\tilde{\theta}(x, z) = \sum_{k=-\infty}^{\infty} (-1)^k k(k+1) x^{k^2} z^{2k};$$

moreover,

$$S_3 = \sum_{k=i_1}^{\infty} q^{k^2/2}\left[|\vartheta|^k + |\vartheta|^{-k}\right],$$
$$S_4 = \tfrac{1}{2} \sum_{k=i_1}^{\infty} k(k+1) q^{k^2/2}\left[|\vartheta|^k + |\vartheta|^{-k}\right]. \tag{2.14}$$

Clearly the sums S_i, $i = 2, 3, 4$, are convergent. Now we wish to find $i_1 = i_1(i)$ such that $S_3 = \mathscr{O}(1/i^2)$, $S_4 = \mathscr{O}(1/i)$ for sufficiently large i's. Owing to (2.12) we get

$$\max\{|\vartheta|, |\vartheta|^{-1}\} = \frac{i}{i-1} q^{-1/2}, \quad \text{on } \mathscr{R}_i.$$

Let i be so large that the inequality

$$q < \mathrm{e}^{-1/\log i} \tag{2.15}$$

holds. Then choosing

$$i_1 = 1 + [2 \log i] > 2 \log i,$$

we find as in the proof of Lemma 2:

$$0 < S_3 < 2q^{(i_1^2 - i_1)/2}\left(\frac{i}{i-1}\right)^{i_1} \frac{1}{1 - q^{i_1} i/(i-1)} < 2\frac{\mathrm{e}}{i^2}\,\mathrm{e}\frac{1}{1 - \mathrm{e}^{-2} i/(i-1)} = \mathscr{O}\left(\frac{1}{i^2}\right),$$

and similarly

$$0 < S_4 < i_1(i_1+1) q^{(i_1^2 - i_1)/2}\left(\frac{i}{i-1}\right)^{i_1} \frac{1}{1 - \mathrm{e}^{-2}(i_1+2)/i_1\; i/(i-1)} = \mathscr{O}\left(\frac{(\log i)^2}{i^2}\right).$$

Hence

$$S_1 = \theta(\sqrt{q}, \sqrt{\vartheta}) - \frac{1}{2i}\tilde{\theta}(\sqrt{q}, \sqrt{\vartheta}) + \mathscr{O}\left(\frac{1}{i^2}\right).$$

Combining this result with relations (2.10), (2.11) and (2.13) we obtain

$$y(z, q) = (-1)^i q^{(i^2-i)/2} \frac{z^i}{i!} \left\{ \theta\left(\sqrt{q}, \sqrt{\vartheta}\right) - \frac{1}{2i} \tilde{\theta}\left(\sqrt{q}, \sqrt{\vartheta}\right) + \mathscr{O}\left(\frac{1}{i^2}\right) \right\}. \tag{2.16}$$

Using the Jacobian identity (1.1) we find for $\vartheta = \mathrm{e}^{\sqrt{-1}\varphi}$, $0 \leqslant \varphi \leqslant 2\pi$, that $\theta(\sqrt{q}, \mathrm{e}^{\sqrt{-1}\varphi}) \neq 0$, hence $|y(z, q)| \neq 0$ for $|z| = iq^{-i+1/2}$, i large, and by the residue theorem the integral

$$N_i = \frac{1}{2\pi\sqrt{-1}} \oint_{|z| = iq^{-i+1/2}} \frac{y'(z, q)}{y(z, q)} \, \mathrm{d}z$$

takes on only integer values; actually N_i is the number of the zeros of $y(z, q)$ counted with multiplicity on the domain $|z| \leqslant iq^{-i+1/2}$. From the differential equation (0.1) we know that $y'(z, q) = -y(qz, q)$ and $qz \in \mathscr{R}_{i-1}$ and by (2.12) the asymptotic formula (2.16) can be applied for $y(qz, q)$ if we substitute $\vartheta i/(i-1)$ for ϑ:

$$y(qz, q) = (-1)^{i-1} q^{(i^2-3i+2)/2} \frac{(qz)^{i-1}}{(i-1)!}$$
$$\times \left\{ \theta\left(\sqrt{q}, \frac{i}{i-1} \mathrm{e}^{\sqrt{-1}\varphi}\right) - \frac{1}{2(i-1)} \tilde{\theta}\left(\sqrt{q}, \frac{i}{i-1} \mathrm{e}^{\sqrt{-1}\varphi}\right) + \mathscr{O}\left(\frac{1}{i^2}\right) \right\},$$

hence we have for N_i:

$$N_i = \frac{i}{2\pi} \int_0^{2\pi} \left\{ 1 + \frac{1}{i-1} \frac{\theta'\left(\sqrt{q}, \mathrm{e}^{\sqrt{-1}\varphi}\right)}{\theta\left(\sqrt{q}, \mathrm{e}^{\sqrt{-1}\varphi}\right)} \mathrm{e}^{\sqrt{-1}\varphi} + \mathscr{O}\left(\frac{1}{i^2}\right) \right\} \mathrm{d}\varphi$$
$$= i + \frac{i}{\sqrt{-1}\,(i-1)2\pi} \int_0^{2\pi} \frac{\mathrm{d}}{\mathrm{d}\varphi} \log \theta\left(\sqrt{q}, \mathrm{e}^{\sqrt{-1}\varphi}\right) \mathrm{d}\varphi + \mathscr{O}\left(\frac{1}{i}\right).$$

From the Jacobian identity (1.1) we obtain

$$\int_0^{2\pi} \frac{\mathrm{d}}{\mathrm{d}\varphi} \log \theta\left(\sqrt{q}, \mathrm{e}^{\sqrt{-1}\varphi}\right) \mathrm{d}\varphi = \sum_{m=1}^{\infty} \left\{ \int_0^{2\pi} \frac{\mathrm{d}}{\mathrm{d}\varphi} \log\left(1 - q^{2m-1} \mathrm{e}^{\sqrt{-1}\varphi}\right) \mathrm{d}\varphi \right.$$
$$\left. + \int_0^{2\pi} \frac{\mathrm{d}}{\mathrm{d}\varphi} \log\left(1 - q^{2m-1} \mathrm{e}^{-\sqrt{-1}\varphi}\right) \mathrm{d}\varphi \right\}.$$

The first integral on the right is 0 because $1 - q^{2m-1}z$ has neither zeros nor poles in $|z| \leqslant 1$, the second integral is also 0 because the function $1 - q^{2m-1}z^{-1}$ has one zero at $z = q^{2m-1}$ and one pole at $z = 0$ both in the unit cirle $|z| < 1$. Consequently $N_i = i + \mathscr{O}(1/i)$, which implies that $N_i = i$ for sufficiently large i's. By the definition of $\mathscr{R}_i$ the number of the zeros on $\mathscr{R}_i$ is equal to $N_i - N_{i-1} = 1$ for sufficiently large i's which is, say, $\hat{t}_i$. This zero must be real because otherwise the complex conjugate $\bar{\hat{t}}_i$ would be another zero in $\mathscr{R}_i$ since $|\hat{t}_i| = |\bar{\hat{t}}_i|$, contradicting the fact that on $\mathscr{R}_i$ there exists only one zero. By (0.4), $\hat{t}_i$ cannot be negative, which completes the proof of Lemma 3. □

Now we can deal with Theorem 1.

Proof of Theorem 1. Suppose the contrary, i.e., $y(z, q_1)$ has a complex-valued zero $t(q_1)$. Since the zeros of $y(z, q)$ depend continuously on the parameter q, the function $t(q)$ can be defined as a zero of $y(z, q)$. Due to Lemma 2 the function $t(q)$ is real for sufficiently small q's, hence there is a value $q^* \in (0, q_1)$ such that $\Im t(q) \neq 0$ on (q^*, q_1) and $\Im t(q^*) = 0$. Hence $t(q^*)$ is real. On the other hand, the complex conjugate $\bar{t}(q)$ is also a zero of $y(z, q)$ for $q \in (q^*, q_1)$, hence the multiplicity of the zero $t(q^*)$ would be at least two. But this contradicts the facts formulated in (0.4). Thus the solution $y(z, q)$ has only real — and consequently positive — zeros. Hence the zero $\hat{t}_i(q)$ mentioned in Lemma 3 equates with $t_i(q)$ for sufficiently large i's. Thereby the proof is complete. □

Now we are in a position to show the asymptotic relation (0.6). We start from the asymptotic form (2.16) of $y(t, q)$ on $\mathscr{R}_i$. By (2.12) we obtain for $t_i(q) = iq^{-i+1/2}\vartheta_i$, when i is sufficiently large, that

$$\frac{i-1}{i}q < \vartheta_i < 1, \qquad \lim_{i\to\infty} \theta\left(\sqrt{q}, \sqrt{\vartheta_i}\right) = 0;$$

hence by (1.1) $\lim_{i\to\infty}\vartheta_i = \sqrt{q}$. Substituting $\vartheta = \sqrt{q}$ in (2.16) and observing the relation

$$\tilde{\vartheta}\left(\sqrt{q}, \sqrt[4]{q}\right) = \sum_{k=-\infty}^{\infty} (-1)^k (k^2 + k) q^{(k^2+k)/2} = 0$$

— because the terms with the indices k and $-k-1$ are cancelling each other — we obtain $\mathscr{O}(1/i^2)$ for the value of the expression in braces on the right-hand side of (2.16). On the other hand, by (1.2) we have

$$\frac{\mathrm{d}}{\mathrm{d}z}\theta\left(\sqrt{q}, z\right)\Big|_{z=\sqrt[4]{q}} = 2q^{-1/4} \prod_{m=1}^{\infty} (1 - q^m)^3 \neq 0,$$

which yields the asymptotic form (0.6).

Proof of Lemma 5. By (0.1), (0.4) and by Theorem 1, $y'(t, q)$ has zeros only at $t_i(q)/q$, $i = 1, 2, \ldots$, and we obtain $t_i/q < t_{i+1}$, which proves the first part of our assertion.

To obtain the upper bound we distinguish two possibilities: either (i) $t_{i+1} \leqslant t_i/q^2$ or (ii) $t_{i+1} > t_i/q^2$.

In case (i) we have by (0.5), (1.5),

$$t_{i+1} = \frac{t_i}{q} + \left(t_{i+1} - \frac{t_i}{q}\right) \leqslant \frac{t_i}{q} + \frac{\pi}{\omega\left(\epsilon t_i/q^2\right)}.$$

In case (ii) let $\tau = \epsilon t_i/q^2$ and consider the functions

$$\tilde{y}(t) = (-1)^i y(t, q), \qquad Y(t) = B\, \mathrm{e}^{\alpha(\tau)t} \sin \omega(\tau)\left(t - \frac{t_i}{q}\right),$$

where B is determined by the requirement $\tilde{y}(t_i/q^2) = Y(t_i/q^2)$. Clearly the function $Y(t)$ is a solution of (1.3), $Y(t) > 0$ for $t \in (t_i/q, t_i/q + \pi/\omega(\tau))$ vanishing at the endpoints of this interval. On the other hand, we have the inequalities for these functions: $Y(t) < \tilde{y}(t)$ for $t_i/q \leqslant t < t_i/q^2$ and for the delays $t - qt = \epsilon t > \tau$ for $(t_i/q^2, \infty)$, hence applying the comparison theorem (see [9, Theorem 34] or also [4]) we obtain $\tilde{y}(t) < Y(t)$ for $t_i/q^2 < t < \min\{t_{i+1}, t_i + \pi/\omega(\tau)\}$, hence $t_{i+1} < t_i + \pi/\omega(\tau)$, which is to be proved. □

Proof of Lemma 7. By [9, Theorem 39] we have $y(t) > e^{-et}$ for $0 < t \leqslant 1/e\epsilon$ and $y(t)$ is strictly decreasing on $[0, t_1]$. In a similar way as in the proof of Lemma 5 we have two possibilities: either (i) $t_1 \leqslant 1/q\ e\epsilon$ or (ii) $t_1 > 1/q\ e\epsilon$. In case (i) the lemma holds. In case (ii) we have to compare $y(t)$ with the solution $Y(t) = B\ e^{\alpha(\tau)t} \sin \omega(\tau)(t - 1/e\epsilon)$ of (1.3) where $\tau = 1/q$ e and B is determined by the requirement $y(1/q\ e\epsilon) = Y(1/q\ e\epsilon)$. Following the pattern of Lemma 5 we get the conclusions on the upper bound of t_1, which proves the lemma. □

Proof of Theorem 11. Let the function $Z(t) = Z(t, q)$ be introduced by

$$Z(t, q) = \sum_{i=1}^{\infty} \frac{1}{t_i(q) - t}, \quad \text{for } 0 \leqslant t < t_1. \tag{2.17}$$

Clearly, by (1.12) the function $Z(t)$ is absolutely convergent on $[0, t_1)$. Then we have by (0.7),

$$y(t, q) = \exp\left\{-\int_0^t Z(w, q)\, dw\right\}, \quad 0 \leqslant t < t_1, \tag{2.18}$$

and by (0.1),

$$Z(t) = \exp\left\{\int_{qt}^t Z(w)\, dw\right\}, \quad 0 \leqslant t < t_1. \tag{2.19}$$

Applying the relations

$$0 < \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right) - \frac{1}{b-a}\int_a^b \frac{ds}{s} = \frac{1}{b-a}\int_a^b \frac{(b-s)^2}{2bs^2}\, ds$$
$$< \frac{1}{2}\int_a^b \frac{ds}{s^2}, \quad 0 < a < b,$$

to $a = t_i - t$, $b = t_{i+1} - t$, $i = 1, 2, \dots$, we obtain by (1.9),

$$\left| Z(t) - \int_{1/e\epsilon}^{\infty} \frac{\gamma_\epsilon(s)}{s - t}\, ds \right| < \frac{1}{2}\frac{1}{t_1 - t} + \frac{1}{2}\int_{t_1 - t}^{\infty} \frac{ds}{s^2} = \frac{1}{t_1 - t}.$$

Then the substitution $\sigma = \epsilon s$; $\tau = \epsilon t$ yields

$$\left| Z\left(\frac{\tau}{\epsilon}\right) - \int_{1/e}^{\infty} \frac{\gamma_\epsilon\left(\frac{\sigma}{\epsilon}\right)}{\sigma - \tau}\, d\sigma \right| < \frac{\epsilon}{\epsilon t_1 - \tau};$$

hence by Corollary 9 and (1.17), (1.18),

$$\lim_{l \to \infty} Z\left(\frac{\tau}{\epsilon_l}, 1 - \epsilon_l\right) = \zeta(\tau).$$

By (2.18) with the substitution $v = \epsilon w$ we obtain

$$\lim_{l\to\infty}\left[y\left(\frac{\tau}{\epsilon_l}, 1-\epsilon_l\right)\right]^{\epsilon_l} = \exp\left(-\int_0^{\tau}\zeta(\vartheta)\,\mathrm{d}\vartheta\right),$$

and similarly for the integral in (2.19),

$$\lim_{l\to\infty}\int_{(1-\epsilon_l)\tau/\epsilon_l}^{\tau/\epsilon_l} Z(w)\,\mathrm{d}w = \tau\zeta(\tau),$$

which completes the proof of our theorem. □

Proof of Theorem 12. By Theorem 11 we have to determine the integral of $\zeta(\tau)$. By (1.19), (1.20) we get

$$\begin{aligned}\int_0^{\tau}\zeta(\vartheta)\,\mathrm{d}\vartheta &= \tau\zeta(\tau) - \int_0^{\tau}[\zeta(\tau)-\zeta(\vartheta)]\,\mathrm{d}\vartheta\\ &= \tau\zeta(\tau) - \int_1^{\zeta(\tau)}\frac{\log\zeta}{\zeta}\,\mathrm{d}\zeta = \log\zeta(\tau) - \tfrac{1}{2}\log^2\zeta(\tau),\end{aligned}$$

which proves the first limit.

Let I be a closed interval in $(1/\mathrm{e}, \infty)$. Since the function $\omega(\tau)$ is continuous and positive for $\tau > 1/\mathrm{e}$, there exists a lower bound ω_0 such that

$$\omega\left(\frac{\tau}{q^2}\right) > \omega_0, \quad \text{for } \tau\in I,$$

provided ϵ is sufficiently small. A consequence of this observation and Lemma 5 is that

$$t_{i+1} - t_i < \frac{\epsilon t_i}{q} + \frac{\pi}{\omega(\epsilon t_i/q^2)} < \frac{\epsilon t_i}{q} + \frac{\pi}{\omega_0} < K, \quad \text{provided } \epsilon t_i \in I, \tag{2.20}$$

where the constant K depends on the interval I, too.

Let Δ be a fixed small number in (0, 1), and define the set $E = E_{\Delta\epsilon}$ by

$$E_{\Delta\epsilon} = \left(t_1 - \frac{\Delta}{\epsilon t_1}, t_1\right) \cup \bigcup_{i=1}^{\infty}\left\{\left[t_i, t_i + \frac{\Delta}{t_{i+1}-t_i}\right) \cup \left(t_{i+1} - \frac{\Delta}{t_{i+1}-t_i}, t_{i+1}\right)\right\}.$$

Then by (0.5), (1.12) and by Lemma 5 we have for the total length of $E_{\Delta\epsilon}$:

$$|E_{\Delta\epsilon}| = \frac{\Delta}{\epsilon t_1} + \sum_{i=1}^{\infty}\frac{2\Delta}{t_{i+1}-t_i} < \frac{\Delta}{\mathrm{e}} + \frac{2q\Delta}{\epsilon} < \frac{2\Delta}{\epsilon}. \tag{2.21}$$

Suppose $\tau\in I$ and $t = \tau/\epsilon \notin E_{\Delta\epsilon}$. Then there is an index k such that $t_k < t < t_{k+1}$ — the possibility $\tau/\epsilon < t_1$ is excluded for sufficiently small ϵ by Corollary 9 and by our assumption on I — and first we estimate the value

$$D = \int_{1/\mathrm{e}\epsilon}^{\infty}\log\left|1-\frac{t}{s}\right|\gamma_\epsilon(s)\,\mathrm{d}s - \sum_{i=1}^{\infty}\log\left|1-\frac{t}{t_i}\right|. \tag{2.22}$$

Let the range of integration be divided into the intervals $[1/e\epsilon, t_k]$, $[t_k, t_{k+1}]$ and $[t_{k+1}, \infty]$; then accordingly let

$$d_1 = \int_{1/e\epsilon}^{t_k} \log\left(\frac{t}{s} - 1\right)\gamma_\epsilon(s)\,\mathrm{d}s - \sum_{i=1}^{k-1} \tfrac{1}{2}\left[\log\left(\frac{t}{t_i} - 1\right) + \log\left(\frac{t}{t_{i+1}} - 1\right)\right],$$

$$d_2 = \frac{1}{t_{k+1} - t_k}\int_{t_k}^{t_{k+1}} \log\left|1 - \frac{t}{s}\right|\mathrm{d}s - \tfrac{1}{2}\log\left(\frac{t}{t_k} - 1\right) - \tfrac{1}{2}\log\left(1 - \frac{t}{t_{k+1}}\right),$$

$$d_3 = \int_{t_{k+1}}^{\infty} \log\left(1 - \frac{t}{s}\right)\gamma_\epsilon(s)\,\mathrm{d}s - \sum_{i=k+1}^{\infty} \tfrac{1}{2}\left[\log\left(1 - \frac{t}{t_i}\right) + \log\left(1 - \frac{t}{t_{i+1}}\right)\right],$$

so we have

$$0 < |D| \leqslant |d_1| + |d_2| + |d_3| + \tfrac{1}{2}\log\left(\frac{t}{t_1} - 1\right). \tag{2.23}$$

Applying Lemma 10 to $\varphi(s) = \log(t - s)$ and $a = t_i$, $b = t_{i+1}$, $i = 1, \ldots, k - 1$, we obtain by (1.14), (2.20),

$$\begin{aligned}
0 &< \int_{1/e\epsilon}^{t_k} \log(t - s)\gamma_\epsilon(s)\,\mathrm{d}s - \sum_{i=1}^{k-1} \tfrac{1}{2}[\log(t - t_i) + \log(t - t_{i+1})] \\
&\leqslant \tfrac{1}{8}\sum_{i=1}^{k-1} (t_{i+1} - t_i)\left[\frac{1}{t - t_{i+1}} - \frac{1}{t - t_i}\right] < \tfrac{1}{8}\pi\sqrt{\varepsilon}\,t\sum_{i=1}^{k-1}\left[\frac{1}{t - t_{i+1}} - \frac{1}{t - t_i}\right] \\
&< \tfrac{1}{8}\pi\sqrt{\epsilon}\,t\frac{1}{t - t_k} \leqslant \tfrac{1}{8}\pi\sqrt{\epsilon}\,t\frac{t_{k+1} - t_k}{\Delta} < \tfrac{1}{8}K\pi\frac{\tau}{\Delta\sqrt{\epsilon}},
\end{aligned}$$

and similarly for $\varphi = \log s$:

$$\begin{aligned}
0 &\leqslant \int_{1/e\epsilon}^{t_k} \log s\,\gamma_\epsilon(s)\,\mathrm{d}s - \sum_{i=1}^{k-1} \tfrac{1}{2}[\log t_i + \log t_{i+1}] \\
&< \tfrac{1}{8}\sum_{i=1}^{k-1} (t_{i+1} - t_i)\left[\frac{1}{t_i} - \frac{1}{t_{i+1}}\right] < \tfrac{1}{8}\pi^2\epsilon t^2\sum_{i=1}^{k-1}\frac{1}{t_i t_{i+1}} < \tfrac{1}{8}\pi^2\epsilon t^2\sum_{i=1}^{\infty}\frac{1}{t_i^2}.
\end{aligned}$$

Now we claim that the sum here equals ϵ. Indeed, comparing the coefficients of the quadratic terms in (0.2) and (0.7), we obtain

$$\sigma_2 = \sum_{1 \leqslant i < j < \infty} \frac{1}{t_i t_j} = \tfrac{1}{2}q,$$

hence by (1.12),

$$\sum_{i=1}^{\infty}\frac{1}{t_i^2} = \left(\sum_{i=1}^{\infty}\frac{1}{t_i}\right)^2 - 2\sigma_2 = 1 - q = \epsilon,$$

as stated. Consequently we have

$$-\tfrac{1}{8}\pi^2\tau^2 < d_1 < \tfrac{1}{8}K\pi\frac{\tau}{\Delta\sqrt{\epsilon}}. \tag{2.24}$$

By similar considerations we get with the choice $a = t_k$, $b = t_{k+1}$,

$$d_2 = \frac{\frac{1}{2}(a+b)-t}{b-a} \log \frac{b-t}{t-a} - 1 - \left\{\frac{1}{b-a}\int_a^b \log s \, \mathrm{d}s - \tfrac{1}{2}[\log a + \log b]\right\},$$

hence by (0.5), (2.20),

$$-1 - \tfrac{1}{8}K^2 \,\mathrm{e}^2\epsilon^2 < d_2 < \log \frac{K}{\sqrt{\Delta}} - 1. \tag{2.25}$$

Finally, we consider d_3. Now $t_i > t = \tau/\epsilon$, $i = k+1, k+2, \dots$, hence by (1.14) we have

$$t_{i+1} - t_i < \frac{\epsilon t_i}{q} + \frac{\pi\epsilon t_i/q^2}{\sqrt{2}\sqrt{1-q^2/(\mathrm{e}\epsilon t_i)}} < c_1(\tau)\epsilon t_i,$$

for sufficiently small ϵ's, and by Lemma 5 applied to $\varphi(s) = \log(1 - t/s)$ we obtain

$$\begin{aligned} 0 < d_3 &< \tfrac{1}{8} \sum_{i=k+1}^{\infty} (t_{i+1} - t_i)\left[\frac{t}{t_i(t_i - t)} - \frac{t}{t_{i+1}(t_{i+1} - t)}\right] \\ &< \tfrac{1}{8}c_1(\tau)\epsilon t \sum_{i=k+1}^{\infty} \frac{t_i + t_{i+1} - t}{t_{i+1}} \frac{t_{i+1} - t_i}{(t_i - t)(t_{i+1} - t)} \\ &< \tfrac{1}{8}c_1(\tau)\tau \sum_{i=k+1}^{\infty} 2\left[\frac{1}{t_i - t} - \frac{1}{t_{i+1} - t}\right] < \tfrac{1}{4}c_1(\tau)\tau\frac{1}{t_{k+1} - t} \leqslant \frac{c_1(\tau)\tau K}{4\Delta}. \end{aligned}$$

Hence by (2.23)–(2.25) the relation $D = \mathscr{O}(1/\Delta\sqrt{\epsilon})$ holds and (2.22) implies

$$\log\left|y\left(\frac{\tau}{\epsilon}\right)\right|^{\epsilon} = \int_{1/e}^{\infty} \log\left|1 - \frac{\tau}{\sigma}\right|\gamma_\epsilon\left(\frac{\sigma}{\epsilon}\right) \mathrm{d}\sigma + \mathscr{O}\left(\frac{\sqrt{\epsilon}}{\Delta}\right), \quad \text{for } \tau \in I,\ \frac{\tau}{\epsilon} \notin E_{\Delta\epsilon}. \tag{2.26}$$

Now the function $(\log|1 - \sigma\tau|)/\sigma$ is not continuous as required in (1.17), but chopping the singularity by

$$L_\delta(\sigma) = \begin{cases} \log \delta, & |\sigma| \leqslant \delta, \\ \log |\sigma|, & |\sigma| > \delta, \end{cases}$$

and observing that

$$\int_{1/e}^{\infty} \left|\log\left|1 - \frac{\tau}{\sigma}\right| - L_\delta\left(1 - \frac{\tau}{\sigma}\right)\right| \frac{\mathrm{d}\sigma}{\sigma} = \mathscr{O}(\delta),$$

we have by (1.17),

$$\lim_{\epsilon \to +0} \int_{1/e}^{\infty} L_\delta\left(1 - \frac{\tau}{\epsilon}\right)\gamma_\epsilon\left(\frac{\tau}{\sigma}\right) \mathrm{d}\sigma = \int_{1/e}^{\infty} L_\delta\left(1 - \frac{\tau}{\sigma}\right)\gamma(\sigma) \,\mathrm{d}\sigma,$$

hence by (1.10), (2.21), (2.26) and by letting $\delta \to +0$,

$$\log\left|y\left(\frac{\tau}{\epsilon}\right)\right|^{\epsilon} \Rightarrow \int_{1/e}^{\infty} \log\left|1 - \frac{\tau}{\sigma}\right|\gamma(\sigma) \,\mathrm{d}\sigma = \Omega(\tau), \quad \text{for } \tau \in I.$$

By this relation and by (1.23),

$$\Omega'(\tau) = -\text{p.v.}\int_{1/e}^{\infty} \frac{\gamma(\sigma)}{\sigma - \tau}\, d\sigma = -\alpha(\tau), \quad \text{for } \tau > \frac{1}{e}.$$

The first part of Theorem 12 — just proved — gives $\Omega(1/e) = -\frac{1}{2}$, hence by (1.6),

$$\Omega(\tau) = -\tfrac{1}{2} + \int_{1/e}^{\tau} \Omega'(\tau)\, d\tau = -\tfrac{1}{2} - \int_{0}^{\mu} \cos\mu \frac{\sin^2\mu - 2\mu \sin\mu \cos\mu + \mu^2}{\sin^3\mu}\, d\mu$$

$$= -\mu^2 - \mu \cot\mu + \frac{1}{2}\frac{\mu^2}{\sin^2\mu},$$

which completes the proof. □

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